A COMBINATORIAL PROOF OF A FORMULA FOR BETTI NUMBERS OF A STACKED POLYTOPE

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Abstract. For a simplicial complex Δ, the graded Betti number $\beta_{i,j}(\mathbf{k}[\Delta])$ of the Stanley-Reisner ring $\mathbf{k}[\Delta]$ over a field $\mathbf{k}$ has a combinatorial interpretation due to Hochster. Terai and Hibi showed that if Δ is the boundary complex of a d-dimensional stacked polytope with n vertices for $d \geq 3$, then $\beta_{k-1,k}(\mathbf{k}[\Delta]) = (k-1)\binom{n-d}{k}$. We prove this combinatorially.

1. Introduction

A *simplicial complex* Δ on a finite set V is a collection of subsets of V satisfying

(1) if $v \in V$ then $\{v\} \in \Delta$,
(2) if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$.

Each element $F \in \Delta$ is called a *face* of Δ. The *dimension* of F is defined by $\dim(F) = |F| - 1$. The *dimension* of Δ is defined by $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$. For a subset $W \subset V$, let Δ_W denote the simplicial complex $\{F \cap W : F \in \Delta\}$ on W.

Let Δ be a simplicial complex on V. Two elements $v, u \in V$ are said to be *connected* if there is a sequence of vertices $v = u_0, u_1, \ldots, u_r = u$ such that $\{u_i, u_{i+1}\} \in \Delta$ for all $i = 0, 1, \ldots, r-1$. A *connected component* C of Δ is a maximal nonempty subset of V such that every two elements of C are connected.

Let $V = \{x_1, x_2, \ldots, x_n\}$ and let R be the polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$ over a fixed field $\mathbf{k}$. Then R is a graded ring with the standard grading $R = \oplus_{i\geq 0} R_i$. Let $R(-j) = \oplus_{i \geq 0}(R(-j))_i$ be the graded module over R with $(R(-j))_i = R_{j+i}$. The *Stanley-Reisner ring* $\mathbf{k}[\Delta]$ of Δ over $\mathbf{k}$ is defined to be R/I_Δ, where I_Δ is the ideal of R generated by the monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta$. A *finite free resolution* of $\mathbf{k}[\Delta]$ is an exact sequence

$$0 \longrightarrow F_r \xrightarrow{\phi_r} F_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} \mathbf{k}[\Delta] \longrightarrow 0\ , \tag{1}$$

where $F_i = \oplus_{j \geq 0} R(-j)^{\beta_{i,j}}$ and each ϕ_i is degree-preserving. A finite free resolution (1) is *minimal* if each $\beta_{i,j}$ is smallest possible. There is a minimal finite free resolution of $\mathbf{k}[\Delta]$ and it is unique up to isomorphism. If (1) is minimal, then the (i,j)*-th graded Betti number* $\beta_{i,j}(\mathbf{k}[\Delta])$ of $\mathbf{k}[\Delta]$ is defined to be $\beta_{i,j}(\mathbf{k}[\Delta]) = \beta_{i,j}$. Hochster's theorem says

$$\beta_{i,j}(\mathbf{k}[\Delta]) = \sum_{\substack{W \subset V \\ |W| = j}} \dim_{\mathbf{k}} \widetilde{H}_{j-i-1}(\Delta_W; \mathbf{k}).$$

We refer the reader to [1, 5] for the details of Betti numbers and Hochster's theorem. Since $\dim_{\mathbf{k}} \widetilde{H}_0(\Delta_W; \mathbf{k})$ is the number of connected components of Δ_W minus 1, we can interpret $\beta_{i-1,i}(\mathbf{k}[\Delta])$ in a purely combinatorial way.

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Definition 1.1. Let Δ be a simplicial complex on a finite nonempty set V. Let k be a nonnegative integer. The *k-th special graded Betti number* $b_k(\Delta)$ of Δ is defined to be

$$b_k(\Delta) = \sum_{\substack{W\subset V\\ |W|=k}} (\mathrm{cc}(\Delta_W) - 1)\,, \tag{2}$$

where $\mathrm{cc}(\Delta_W)$ denotes the number of connected components of Δ_W.

Note that since there is no connected component in $\Delta_\emptyset = \{\emptyset\}$, we have $b_0(\Delta) = -1$. If $k > |V|$, then $b_k(\Delta) = 0$ because there is nothing in the sum in (2). Thus we have

$$b_k(\Delta) = \begin{cases} \beta_{k-1,k}(\mathbf{k}[\Delta]), & \text{if } k \geq 1, \\ -1, & \text{if } k = 0. \end{cases}$$

We refer the reader to [7] for the basic notions of convex polytopes. Let P be a simplicial polytope with vertex set V. The *boundary complex* $\Delta(P)$ is the simplicial complex Δ on V such that $F \in \Delta$ for some $F \subset V$ if and only if $F \neq V$ and the convex hull of F is a face of P. Note that if the dimension of P is d, then $\dim(\Delta(P)) = d - 1$.

For a d-dimensional simplicial polytope P, we can attach a d-dimensional simplex to a facet of P. A *stacked polytope* is a simplicial polytope obtained in this way starting with a d-dimensional simplex.

Let P be a d-dimensional stacked polytope with n vertices. Hibi and Terai [6] showed that $\beta_{i,j}(\mathbf{k}[\Delta(P)]) = 0$ unless $i = j-1$ or $i = j-d+1$. Since $\beta_{i-1,i}(\mathbf{k}[\Delta(P)]) = \beta_{n-i-d+1,n-i}(\mathbf{k}[\Delta(P)])$, it is sufficient to determine $\beta_{i-1,i}(\mathbf{k}[\Delta(P)])$ to find all $\beta_{i,j}(\mathbf{k}[\Delta(P)])$. In the same paper, they found the following formula for $\beta_{k-1,k}(\mathbf{k}[\Delta(P)])$:

$$\beta_{k-1,k}(\mathbf{k}[\Delta(P)]) = (k-1)\binom{n-d}{k}. \tag{3}$$

Herzog and Li Marzi [4] gave another proof of (3).

The main purpose of this paper is to prove (3) combinatorially. In the meanwhile, we get as corollaries the results of Bruns and Hibi [2] : a formula of $b_k(\Delta)$ if Δ is a tree (or a cycle) considered as a 1-dimensional simplicial complex.

2. Definition of t-connected sum

In this section we define a t-connected sum of simplicial complexes, which gives another equivalent definition of the boundary complex of a stacked polytope. See [3] for the details of connected sums. And then, we extend the definition of t-connected sum to graphs, which has less restrictions on the construction. Every graph in this paper is simple.

2.1. A t-connected sum of simplicial complexes.

Let V and V' be finite sets. A *relabeling* is a bijection $\sigma : V \to V'$. If Δ is a simplicial complex on V, then $\sigma(\Delta) = \{\sigma(F) : F \in \Delta\}$ is a simplicial complex on V'.

Definition 2.1. Let Δ_1 and Δ_2 be simplicial complexes on V_1 and V_2 respectively. Let $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$ be maximal faces with $|F_1| = |F_2|$. Let V_2' be a finite set and $\sigma : V_2 \to V_2'$ a relabeling such that $V_1 \cap V_2' = F_1$ and $\sigma(F_2) = F_1$. Then the *connected sum* $\Delta_1 \#_\sigma^{F_1,F_2} \Delta_2$ of Δ_1 and Δ_2 with respect to (F_1, F_2, σ) is the simplicial complex $(\Delta_1 \cup \sigma(\Delta_2)) \setminus \{F_1\}$ on $V_1 \cup V_2'$. If $\Delta = \Delta_1 \#_\sigma^{F_1,F_2} \Delta_2$ and $|F_1| = |F_2| = t$, then we say that Δ is a *t-connected sum* of Δ_1 and Δ_2.

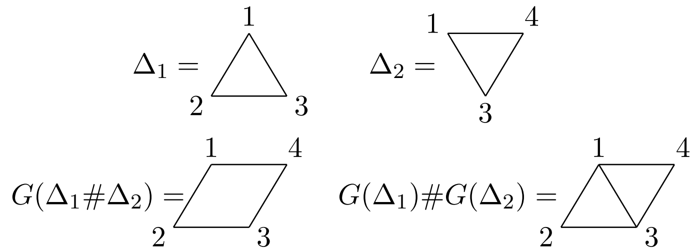

FIGURE 1. The 1-skeleton of a 2-connected sum of Δ_1 and Δ_2 is not a 2-connected sum of $G(\Delta_1)$ and $G(\Delta_2)$.

Note that if Δ_1 and Δ_2 are $(d-1)$-dimensional pure simplicial complexes, i.e. the dimension of each maximal face is $d-1$, then we can only define a d-connected sum of them.

Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be simplicial complexes. A simplicial complex Δ is said to be a t-connected sum of $\Delta_1, \Delta_2, \ldots, \Delta_n$ if there is a sequence of simplicial complexes $\Delta'_1, \Delta'_2, \ldots, \Delta'_n$ such that $\Delta'_1 = \Delta_1$, Δ'_i is a t-connected sum of Δ'_{i-1} and Δ_i for $i = 2, 3, \ldots, n$, and $\Delta'_n = \Delta$.

2.2. **A t-connected sum of graphs.** Let G be a graph with vertex set V and edge set E. Let $W \subset V$. Then the *induced subgraph* $G|_W$ of G with respect to W is the graph with vertex set W and edge set $\{\{x, y\} \in E : x, y \in W\}$. Let

$$b_k(G) = \sum_{\substack{W \subset V \\ |W| = k}} \left(\operatorname{cc}(G|_W) - 1\right),$$

where $\operatorname{cc}(G|_W)$ denotes the number of connected components of $G|_W$.

Let Δ be a simplicial complex on V. The 1-*skeleton* $G(\Delta)$ of Δ is the graph with vertex set V and edge set $E = \{F \in \Delta : |F| = 2\}$. By definition, the connected components of Δ_W and $G(\Delta)|_W$ are identical for all $W \subset V$. Thus $b_k(\Delta) = b_k(G(\Delta))$.

Now we define a t-connected sum of two graphs.

Definition 2.2. Let G_1 and G_2 be graphs with vertex sets V_1 and V_2, and edge sets E_1 and E_2 respectively. Let $F_1 \subset V_1$ and $F_2 \subset V_2$ be sets of vertices such that $|F_1| = |F_2|$, and $G_1|_{F_1}$ and $G_2|_{F_2}$ are complete graphs. Let V'_2 be a finite set and $\sigma : V_2 \to V'_2$ a relabeling such that $V_1 \cap V'_2 = F_1$ and $\sigma(F_2) = F_1$. Then the *connected sum* $G_1 \#_\sigma^{F_1, F_2} G_2$ of G_1 and G_2 with respect to (F_1, F_2, σ) is the graph with vertex set $V_1 \cup V'_2$ and edge set $E_1 \cup \sigma(E_2)$, where $\sigma(E_2) = \{\{\sigma(x), \sigma(y)\} : \{x, y\} \in E_2\}$. If $G = G_1 \#_\sigma^{F_1, F_2} G_2$ and $|F_1| = |F_2| = t$, then we say that G is a t-*connected sum* of G_1 and G_2.

Note that in contrary to the definition of t-connected sum of simplicial complexes, it is not required that F_1 and F_2 are maximal, and we do not remove any element in $E_1 \cup \sigma(E_2)$. We define a t-connected sum of $G_1, G_2, \ldots, G_n$ as we did for simplicial complexes.

It is easy to see that, if $|F_1| = |F_2| \geq 3$ then $G(\Delta_1 \#_\sigma^{F_1, F_2} \Delta_2) = G(\Delta_1) \#_\sigma^{F_1, F_2} G(\Delta_2)$. Thus we get the following proposition.

Proposition 2.3. *For $t \geq 3$, if Δ is a t-connected sum of $\Delta_1, \Delta_2, \ldots, \Delta_n$, then $G(\Delta)$ is a t-connected sum of $G(\Delta_1), G(\Delta_2), \ldots, G(\Delta_n)$.*

Note that Proposition 2.3 is not true if $t = 2$ as the following example shows.

Example 2.4. Let $\Delta_1 = \{12, 23, 13\}$ and $\Delta_2 = \{13, 34, 14\}$ be simplicial complexes on $V_1 = \{1, 2, 3\}$ and $V_2 = \{1, 3, 4\}$. Here 12 means the set $\{1, 2\}$. Let $F_1 = F_2 = \{1, 3\}$ and let σ

be the identity map from V_2 to itself. Then the edge set of $G(\Delta_1 \#_\sigma^{F_1,F_2} \Delta_2)$ is $\{12, 23, 34, 14\}$, but the edge set of $G(\Delta_1) \#_\sigma^{F_1,F_2} G(\Delta_2)$ is $\{12, 23, 34, 14, 13\}$. See Figure 1.

3. Main results

In this section we find a formula of $b_k(G)$ for a graph G which is a t-connected sum of two graphs. To do this let us introduce the following notation. For a graph G with vertex set V, let

$$c_k(G) = \sum_{\substack{W \subset V \\ |W| = k}} \mathrm{cc}(G|_W).$$

Note that $c_k(G) = b_k(G) + \binom{|V|}{k}$.

Lemma 3.1. *Let G_1 and G_2 be graphs with n_1 and n_2 vertices respectively. Let t be a positive integer and let G be a t-connected sum of G_1 and G_2. Then*

$$\begin{aligned} c_k(G) = &\sum_{i=0}^{k} \left(c_i(G_1)\binom{n_2 - t}{k - i} + c_i(G_2)\binom{n_1 - t}{k - i} \right) \\ &- \binom{n_1 + n_2 - t}{k} + \binom{n_1 + n_2 - 2t}{k}. \end{aligned}$$

Proof. Let V_1 (resp. V_2) be the vertex set of G_1 (resp. G_2). We have $G = G_1 \#_\sigma^{F_1,F_2} G_2$ for some $F_1 \subset V_1$, $F_2 \subset V_2$, a vertex set V_2' and a relabeling $\sigma : V_1 \to V_2'$ such that $V_1 \cap V_2' = F_1$, $\sigma(F_2) = F_1$, and $G_1|_{F_1}$ and $G_2|_{F_2}$ are complete graphs on t vertices.

Let A be the set of pairs (C, W) such that $W \subset V_1 \cup V_2'$, $|W| = k$ and C is a connected component of $G|_W$. Let

$$A_1 = \{(C, W) \in A : C \cap V_1 \neq \emptyset\}, \quad A_2 = \{(C, W) \in A : C \cap V_2' \neq \emptyset\}.$$

Then $c_k(G) = |A| = |A_1| + |A_2| - |A_1 \cap A_2|$. It is sufficient to show that $|A_1| = \sum_{i=0}^{k} c_i(G_1)\binom{n_2-t}{k-i}$, $|A_2| = \sum_{i=0}^{k} c_i(G_2)\binom{n_1-t}{k-i}$ and $|A_1 \cap A_2| = \binom{n_1+n_2-t}{k} - \binom{n_1+n_2-2t}{k}$.

Let B_1 be the set of triples (C_1, W_1, X) such that $W_1 \subset V_1$, $X \subset V_2' \setminus V_1$, $|X| + |W_1| = k$ and C_1 is a connected component of $G_1|_{W_1}$. Let $\phi_1 : A_1 \to B_1$ be the map defined by $\phi_1(C, W) = (C \cap V_1, W \cap V_1, W \setminus V_1)$. Then ϕ_1 has the inverse map defined as follows. For a triple $(C_1, W_1, X) \in B_1$, $\phi_1^{-1}(C_1, W_1, X) = (C, W)$, where $W = W_1 \cup X$ and C is the connected component of $G|_W$ containing C_1. Thus ϕ_1 is a bijection and we get $|A_1| = |B_1| = \sum_{i=0}^{k} c_i(G_1)\binom{n_2-t}{k-i}$. Similarly we get $|A_2| = \sum_{i=0}^{k} c_i(G_2)\binom{n_1-t}{k-i}$.

Now let $B = \{W \subset V_1 \cup V_2' : W \cap F_1 \neq \emptyset\}$. Let $\psi : A_1 \cap A_2 \to B$ be the map defined by $\psi(C, W) = W$. We have the inverse map ψ^{-1} as follows. For $W \in B$, $\psi^{-1}(W) = (C, W)$, where C is the connected component of $G|_W$ containing $W \cap F_1$, which is guaranteed to exist since $G|_{F_1} = G_1|_{F_1}$ is a complete graph. Thus ψ is a bijection, and we get $|A_1 \cap A_2| = |B| = \binom{n_1+n_2-t}{k} - \binom{n_1+n_2-2t}{k}$. ∎

Theorem 3.2. *Let G_1 and G_2 be graphs with n_1 and n_2 vertices respectively. Let t be a positive integer and let G be a t-connected sum of G_1 and G_2. Then*

$$b_k(G) = \sum_{i=0}^{k} \left(b_i(G_1)\binom{n_2 - t}{k - i} + b_i(G_2)\binom{n_1 - t}{k - i} \right) + \binom{n_1 + n_2 - 2t}{k}.$$

Proof. Since $c_k(G) = b_k(G) + \binom{n_1+n_2-t}{k}$, $c_i(G_1) = b_i(G_1) + \binom{n_1}{i}$ and $c_i(G_2) = b_i(G_2) + \binom{n_2}{i}$, by Lemma 3.1, it is sufficient to show that

$$2\binom{n_1+n_2-t}{k} = \sum_{i=0}^{k}\left(\binom{n_1}{i}\binom{n_2-t}{k-i} + \binom{n_2}{i}\binom{n_1-t}{k-i}\right),$$

which is immediate from the identity $\sum_{i=0}^{k}\binom{a}{i}\binom{b}{k-i} = \binom{a+b}{k}$. □

Recall that a t-connected sum G of two graphs depends on the choice of vertices of each graph and the identification of the chosen vertices. However, Theorem 3.2 says that $b_k(G)$ does not depend on them. Thus we get the following important property of a t-connected sum of graphs.

Corollary 3.3. *Let t be a positive integer and let G be a t-connected sum of graphs $G_1, G_2, \ldots, G_n$. If H is also a t-connected sum of $G_1, G_2, \ldots, G_n$, then $b_k(G) = b_k(H)$ for all k.*

Using Proposition 2.3, we get a formula for the special graded Betti number of a t-connected sum of two simplicial complexes for $t \geq 3$.

Corollary 3.4. *Let Δ_1 and Δ_2 be simplicial complexes on V_1 and V_2 respectively with $|V_1| = n_1$ and $|V_2| = n_2$. Let t be a positive integer and let Δ be a t-connected sum of Δ_1 and Δ_2. If $t \geq 3$ then*

$$b_k(\Delta) = \sum_{i=0}^{k}\left(b_i(\Delta_1)\binom{n_2-t}{k-i} + b_i(\Delta_2)\binom{n_1-t}{k-i}\right) + \binom{n_1+n_2-2t}{k}.$$

For an integer n, let K_n denote a complete graph with n vertices.

Let G be a graph with vertex set V. If H is a t-connected sum of G and K_{t+1} then H is a graph obtained from G by adding a new vertex v connected to all vertices in W for some $W \subset V$ such that $G|_W$ is isomorphic to K_t. Thus H is determined by choosing such a subset $W \subset V$. Using this observation, we get the following lemma.

Theorem 3.5. *Let t be a positive integer. Let G be a t-connected sum of n K_{t+1}'s. Then*

$$b_k(G) = (k-1)\binom{n}{k}.$$

Proof. We construct a sequence of graphs $H_1, \ldots, H_n$ as follows. Let H_1 be the complete graph with vertex set $\{v_1, v_2, \ldots, v_{t+1}\}$. For $i \geq 2$, let H_i be the graph obtained from H_{i-1} by adding a new vertex v_{t+i} connected to all vertices in $\{v_1, v_2, \ldots, v_t\}$. Then H_n is a t-connected sum of n K_{t+1}'s, and we have $b_k(G) = b_k(H_n)$ by Corollary 3.3. In H_n, the vertex v_i is connected to all the other vertices for $i \leq t$, and v_j and $v_{j'}$ are not connected to each other for all $t+1 \leq j, j' \leq t+n$. Thus $b_k(H_n) = (k-1)\binom{n}{k}$. □

Observe that every tree with $n+1$ vertices is a 1-connected sum of n K_2's. Thus we get the following nontrivial property of trees which was observed by Bruns and Hibi [2].

Corollary 3.6. [2, Example 2.1. (b)] *Let T be a tree with $n+1$ vertices. Then $b_k(T)$ does not depend on the specific tree T. We have*

$$b_k(T) = (k-1)\binom{n}{k}.$$

Corollary 3.7. [2, Example 2.1. (c)] *Let G be an n-gon. If $k = n$ then $b_k(G) = 0$; otherwise,*

$$b_k(G) = \frac{n(k-1)}{n-k}\binom{n-2}{k}.$$

Proof. It is clear for $k = n$. Assume $k < n$. Let $V = \{v_1, \dots, v_n\}$ be the vertex set of G. Then

$$\begin{aligned}(n-k)\cdot b_k(G) &= \sum_{\substack{W\subset V\\ |W|=k}} (\mathrm{cc}(G|_W) - 1) \sum_{v\in V\setminus W} 1\\ &= \sum_{v\in V}\sum_{\substack{W\subset V\setminus\{v\}\\ |W|=k}} (\mathrm{cc}(G|_W) - 1)\\ &= \sum_{v\in V} b_k(G|_{V\setminus\{v\}}).\end{aligned}$$

Since each $G|_{V\setminus\{v\}}$ is a tree with $n-1$ vertices, we are done by Corollary 3.6. □

Remark 3.8. Bruns and Hibi [2] obtained Corollary 3.6 and Corollary 3.7 by showing that if Δ is a tree (or an n-gon), considered as a 1-dimensional simplicial complex, then $\mathbf{k}[\Delta]$ has a pure resolution. Since $\mathbf{k}[\Delta]$ is Cohen-Macaulay and it has a pure resolution, the Betti numbers are determined by its type (c.f. [1]).

Now we can prove (3). Note that, for $d \geq 3$, if P is a d-dimensional simplicial polytope and Q is a simplicial polytope obtained from P by attaching a d-dimensional simplex S to a facet of P, then $\Delta(Q)$ is a d-connected sum of $\Delta(P)$ and $\Delta(S)$, and thus the 1-skeleton $G(\Delta(Q))$ is a d-connected sum of $G(\Delta(P))$ and K_{d+1}. Hence the 1-skeleton of the boundary complex of a d-dimensional stacked polytope is a d-connected sum of K_{d+1}'s.

Theorem 3.9. *Let P be a d-dimensional stacked polytope with n vertices. If $d \geq 3$, then*

$$b_k(\Delta(P)) = (k-1)\binom{n-d}{k}.$$

If $d = 2$, then

$$b_k(\Delta(P)) = \begin{cases} 0, & \text{if } k = n,\\ \frac{n(k-1)}{n-k}\binom{n-2}{k}, & \text{otherwise.}\end{cases}$$

Proof. Assume $d \geq 3$. Then the 1-skeleton $G(\Delta(P))$ is a d-connected sum of $n-d$ K_{d+1}'s. Thus by Theorem 3.5, we get $b_k(\Delta(P)) = b_k(G(\Delta(P))) = (k-1)\binom{n-d}{k}$.

Now assume $d = 2$. Then $G(\Delta(P))$ is an n-gon. Thus by Corollary 3.7 we are done. □

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